

# Reexamination of Entanglement of Superpositions

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We find tight lower and upper bounds on the entanglement of a superposition of two bipartite states in terms of the entanglement of the two states constituting the superposition. Our upper bound is dramatically tighter than the one presented in Phys. Rev. Lett **97**, 100502 (2006) and our lower bound can be used to provide lower bounds on different measures of entanglement such as the entanglement of formation and the entanglement of subspaces. We also find that in the case in which the two states are one-sided orthogonal, the entanglement of the superposition state can be expressed explicitly in terms of the entanglement of the two states in the superposition.

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In a recent paper by Linden, Popescu and Smolin (LPS) [1], the authors raised the following question: Given a *bipartite* state  $|\Gamma\rangle$ , and given a certain decomposition of it as a superposition of two bipartite states

$$|\Gamma\rangle = \alpha|\Psi\rangle + \beta|\Phi\rangle; \quad |\alpha|^2 + |\beta|^2 = 1$$

what is the relation between the entanglement of  $|\Gamma\rangle$  and the entanglement of  $|\Psi\rangle$  and  $|\Phi\rangle$ ? It is somewhat surprising that very little is known about this basic question given how important entanglement is to quantum mechanics and how in bipartite settings superposition is almost a synonymous term to entanglement. Perhaps one of the reasons for that is that the entanglement of  $|\Gamma\rangle$ , depends also on the coherence between the two terms in the decomposition, and therefore in general it does not depend only on the entanglement of  $|\Psi\rangle$  and  $|\Phi\rangle$ . This can be seen most clearly in the Bell state example with  $|\Psi\rangle = |00\rangle$ ,  $|\Phi\rangle = |11\rangle$ , and  $\alpha = \beta = 1/\sqrt{2}$ . Nevertheless, in [1] the authors have found an upper bound (dubbed here the LPS bound) on the entropy of entanglement of  $|\Gamma\rangle$  given in terms of the entanglement of  $|\Psi\rangle$  and  $|\Phi\rangle$ . Subsequently, several authors generalised this result to include different measures of entanglement [2, 3, 4], entanglement of superpositions of multipartite states [5, 6] and entanglement superpositions of more than two states [7].

In this Letter we show that the LPS upper bound is not tight and can be improved dramatically if one includes two factors. The first one is based on a generalization of biorthogonal states to include one-sided orthogonal bipartite states. This factor leads to a slight improvement of the LPS bound. The second more important factor that leads to a dramatic improvement is based on the relation between different convex decompositions of a density matrix. We find that unless  $E(\Psi) = E(\Phi)$  and  $|\alpha| = |\beta|$  our bound is strictly tighter and in general, in the limit of large dimensions can be arbitrarily tighter. Our method also enables us to find a tight lower bound that depends only on  $E(\Psi)$ ,  $E(\Phi)$ ,  $|\alpha|$  and  $|\beta|$ .

We start with a definition of one-sided orthogonal bi-

partite states and a simple improvement of the LPS bound.

**Definition 1.** *One sided orthogonal bipartite states:* Two bipartite states  $|\Psi\rangle_{AB}$  and  $|\Phi\rangle_{AB}$  are one sided orthogonal if

$$\text{Tr}_B [\text{Tr}_A (|\Psi\rangle\langle\Psi|) \text{Tr}_A (|\Phi\rangle\langle\Phi|)] = 0 \quad (1)$$

or

$$\text{Tr}_A [\text{Tr}_B (|\Psi\rangle\langle\Psi|) \text{Tr}_B (|\Phi\rangle\langle\Phi|)] = 0. \quad (2)$$

Note that one sided orthogonal states are orthogonal but not necessarily biorthogonal (i.e. for one sided orthogonal states in general only one of the two equations above is satisfied). In the following, with out loss of generality, we assume that one-sided orthogonal states satisfy Eq. (1) but not necessarily Eq. (2).

**Lemma 1.** *Up to local unitary transformations, one-sided orthogonal states can be written as:*

$$|\Psi\rangle = \sum_{i=1}^{d_1} \sqrt{p_i} |u_i\rangle_A |i\rangle_B \quad \text{and} \quad |\Phi\rangle = \sum_{i=1}^{d_2} \sqrt{q_i} |v_i\rangle_A |i+d_1\rangle_B, \quad (3)$$

where  $\{p_i\}$  and  $\{q_i\}$  are two sets of positive numbers that sums to one, and  $\{|u_i\rangle_A\}$  and  $\{|v_i\rangle_A\}$  are two sets of orthonormal states.

Note that if  ${}_A\langle v_{i'} | u_i \rangle_A = 0$  for all  $i = 1, 2, \dots, d_1$  and  $i' = 1, 2, \dots, d_2$  then the states are biorthogonal.

*Proof.* Due to the Shmidt decomposition we have

$$|\Psi\rangle = \sum_{i=1}^{d_1} \sqrt{p_i} |u_i\rangle_A |u_i\rangle_B \quad \text{and} \quad |\Phi\rangle = \sum_{i=1}^{d_2} \sqrt{q_i} |v_i\rangle_A |v_i\rangle_B,$$

where  $\{|u_i\rangle\}$  and  $\{|v_i\rangle\}$  are sets of orthonormal states, and  $\{p_i\}$  and  $\{q_i\}$  are two sets of positive numbers that sums to one. Since we assume that the states satisfy Eq. (1), we get that  ${}_B\langle v_{i'} | u_i \rangle_B = 0$  for all  $i =$

$1, 2, \dots, d_1$  and  $i' = 1, 2, \dots, d_2$ . Thus, we can define the set  $\{|i\rangle_B\}_{i=1}^{d_1+d_2}$ , where  $|i\rangle_B = |u_i\rangle_B$  for  $1 \leq i \leq d_1$  and  $|i\rangle_B = |v_{i-d_1}\rangle_B$  for  $d_1 + 1 \leq i \leq d_1 + d_2$ . With these notations we obtain Eq. (3).  $\square$

**Theorem 2.** *Given  $|\Psi\rangle$  and  $|\Phi\rangle$  one-sided orthogonal, and  $|\alpha|^2 + |\beta|^2 = 1$ , the entanglement of the superposition  $|\Gamma\rangle = \alpha|\Psi\rangle + \beta|\Phi\rangle$  obeys*

$$E(\Gamma) = S(\rho^A) = |\alpha|^2 E(\Psi) + |\beta|^2 E(\Phi) + S(\rho^{AB}) - |S(\rho^A) - S(\rho^B)| \quad (4)$$

where  $\rho^{AB} = |\alpha|^2 |\Psi\rangle\langle\Psi| + |\beta|^2 |\Phi\rangle\langle\Phi|$  and  $\rho^A$  and  $\rho^B$  are obtained by tracing  $\rho^{AB}$  over  $B$  and  $A$ , respectively.

Few remarks are in order. First, since  $|\Phi\rangle$  and  $|\Psi\rangle$  are orthogonal we have  $S(\rho^{AB}) = h_2(|\alpha|^2)$  where  $h_2(x) = -x \log x - (1-x) \log(1-x)$  is the binary entropy function. Second, for biorthogonal states  $S(\rho^A) = S(\rho^B)$  and so we obtain the formula given in [1] for that case. Third, note that the right hand side of Eq. (4) depends only on quantities with no coherence between  $|\Phi\rangle$  and  $|\Psi\rangle$ . Forth, from the triangle inequality of the von-Neumann entropy (i.e. the Araki-Lieb inequality) we have  $S(\rho^{AB}) \geq |S(\rho^A) - S(\rho^B)|$ .

*Proof.* Due to Lemma 1 we have  $\text{Tr}_B |\Gamma\rangle\langle\Gamma| = \rho^A$ . Hence,  $E(\Gamma) = S(\rho^A)$ . Further, from Lemma 1 it follows that the eigenvalues of  $\rho^B \equiv |\alpha|^2 \text{Tr}_A |\Psi\rangle\langle\Psi| + |\beta|^2 \text{Tr}_A |\Phi\rangle\langle\Phi|$  are  $\{|\alpha|^2 p_i\}$  and  $\{|\beta|^2 q_j\}$  for  $i = 1, \dots, d_1$  and  $j = 1, \dots, d_2$ . Thus,  $S(\rho^B) = |\alpha|^2 E(\Psi) + |\beta|^2 E(\Phi) + S(\rho^{AB})$ . This also implies that for one-sided orthogonal states satisfying Eq. (1),  $S(\rho^B) \geq S(\rho^A)$ . This completes the proof.  $\square$

*Example 1.* Consider the one-sided orthogonal states

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \text{ and } |\Phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|2\rangle + |1\rangle|3\rangle).$$

Clearly,  $E(\Psi) = E(\Phi) = 1$ . Now, it is also easy to check that  $E(\Gamma) = 1$  for *any* coherent superposition  $|\Gamma\rangle = \alpha|\Psi\rangle + \beta|\Phi\rangle$ . Therefore, the left hand side of Eq. (4) is equal to 1. One can also check that  $S(\rho^A) = 1$  whereas  $S(\rho^B) = 1 + h_2(|\alpha|^2)$ . Thus, the right hand side of Eq. (4) is also equal to 1.

Before we present our two main results (Theorem 3 and Theorem 4), we briefly review the main theorem in [1] and provide a slight improvement of it. The authors in [1] have used two properties of the von-Neumann entropy:

$$|\alpha|^2 S(\sigma_1) + |\beta|^2 S(\sigma_2) \leq S(|\alpha|^2 \sigma_1 + |\beta|^2 \sigma_2) \quad (5)$$

and

$$S(|\alpha|^2 \sigma_1 + |\beta|^2 \sigma_2) \leq |\alpha|^2 S(\sigma_1) + |\beta|^2 S(\sigma_2) + h_2(|\alpha|^2). \quad (6)$$

Now, consider  $\rho^{AB}$  and  $\rho^A$  as defined in Theorem 1, except that now  $\Psi$  and  $\Phi$  are not necessarily orthogonal. We can write  $\rho^A = |\alpha|^2 \sigma_1 + |\beta|^2 \sigma_2$ , where  $\sigma_1 = \text{Tr}_B |\Psi\rangle\langle\Psi|$  and  $\sigma_2 = \text{Tr}_B |\Phi\rangle\langle\Phi|$ . Using Eq. (6) we get,

$$S(\rho^A) \leq |\alpha|^2 E(\Psi) + |\beta|^2 E(\Phi) + h_2(|\alpha|^2). \quad (7)$$

Next, the state  $\rho^A$  can also be decomposed as  $\rho^A = (n_+/2)\sigma_+ + (n_-/2)\sigma_-$ , where  $\sigma_{\pm} \equiv (1/n_{\pm}) \text{Tr}_B |\Gamma_{\pm}\rangle\langle\Gamma_{\pm}|$ ,  $|\Gamma_{\pm}\rangle = \alpha|\Psi\rangle \pm \beta|\Phi\rangle$  and  $n_{\pm} = \|\Gamma_{\pm}\|^2$ . Thus, from Eq.(5) we get

$$\frac{n_+}{2} E(\Gamma_+) + \frac{n_-}{2} E(\Gamma_-) \leq S(\rho^A), \quad (8)$$

(the notation  $E(\Gamma_{\pm})$  refers to the entanglement of the normalized states  $(1/\sqrt{n_{\pm}})|\Gamma_{\pm}\rangle$ ). Combining Eq. (7) with Eq. (8) and using the fact that  $E(\Gamma_-) \geq 0$  one obtains the LPS bound:

$$\|\Gamma\|^2 E(\Gamma) \leq 2(|\alpha|^2 E(\Psi) + |\beta|^2 E(\Phi) + h_2(|\alpha|^2)),$$

(here  $\Gamma \equiv \Gamma_+$ ).

We now present a simple improvement of the above LPS bound.

**Theorem 3.** *Let  $|\Psi\rangle$  and  $|\Phi\rangle$  be two bipartite states, and let  $\alpha, \beta \in \mathbb{C}$  such that  $|\alpha|^2 + |\beta|^2 = 1$ . Then,*

$$\|\alpha|\Psi\rangle + \beta|\Phi\rangle\|^2 E(\alpha|\Psi\rangle + \beta|\Phi\rangle) \leq 2 \left[ |\alpha|^2 E(\Psi) + |\beta|^2 E(\Phi) + h_2(|\alpha|^2) - |S(\rho^A) - S(\rho^B)| \right].$$

*Proof.* To prove it we improve the bounds given in Eqs. (7,8). Eq. (8) can be slightly improved by writing

$$\frac{n_+}{2} E(\Gamma_+) + \frac{n_-}{2} E(\Gamma_-) \leq \min\{S(\rho^A), S(\rho^B)\}, \quad (9)$$

since one can repeat the same arguments that led to Eq. (8) with  $\rho^B$  instead of  $\rho^A$ . In the same way, Eq. (7) can be improved to the following one:

$$\max\{S(\rho^A), S(\rho^B)\} \leq |\alpha|^2 E(\Psi) + |\beta|^2 E(\Phi) + h_2(|\alpha|^2).$$

Thus,

$$\min\{S(\rho^A), S(\rho^B)\} \leq |\alpha|^2 E(\Psi) + |\beta|^2 E(\Phi) + h_2(|\alpha|^2) - |S(\rho^A) - S(\rho^B)|. \quad (10)$$

The combination of Eq.(9) and Eq. (10) provides the proof for Theorem 2.  $\square$

*Example 2.* Consider the following example when Alice and Bob have Hilbert spaces of dimensions 3 and 4, respectively:

$$|\Psi\rangle = \sqrt{\frac{1}{2}}|00\rangle + \frac{1}{2}|11\rangle + \frac{1}{2}|22\rangle$$

$$|\Psi\rangle = \sqrt{\frac{1}{2}}|03\rangle + \frac{1}{2}|11\rangle + \frac{1}{2}|22\rangle$$

$$\alpha = \beta = \frac{1}{\sqrt{2}}.$$

The entanglement of  $|\Psi\rangle$  and  $|\Phi\rangle$  is  $3/2$ , and the entanglement of  $\alpha|\Psi\rangle + \beta|\Phi\rangle$  is  $\log 3 > 3/2$ . Since  $\|\alpha|\Psi\rangle + \beta|\Phi\rangle\| = \sqrt{3/2}$ , the LPS upper bound is  $E(\alpha|\Psi\rangle + \beta|\Phi\rangle) \leq 5\sqrt{2/3}$ . Since  $S(\rho^A) = 3/2$  and  $S(\rho^B) = 2$ , our bound is  $E(\alpha|\Psi\rangle + \beta|\Phi\rangle) \leq 4\sqrt{2/3}$  (i.e. an improvement by almost 1 ebit).

The bound in theorem 2 provides an improvement of the LPS bound. However, since  $h_2(|\alpha|^2) \geq |S(\rho^A) - S(\rho^B)|$  our bound is smaller by no more than 2 ebits from the LPS bound. We now ready to introduce first of our two main results which provides a new upper bound that can be arbitrarily smaller than the LPS bound.

**Theorem 4.** *Let  $|\Psi\rangle$  and  $|\Phi\rangle$  be two bipartite states, and let  $\alpha, \beta \in \mathbb{C}$  such that  $|\alpha|^2 + |\beta|^2 = 1$ . Then,*

$$\|\alpha|\Psi\rangle + \beta|\Phi\rangle\|^2 E(\alpha|\Psi\rangle + \beta|\Phi\rangle) \leq f(t), \quad (11)$$

for all  $0 \leq t \leq 1$ , where

$$f(t) = \frac{t|\beta|^2 + (1-t)|\alpha|^2}{t(1-t)} \left[ tE(|\Psi\rangle) + (1-t)E(|\Phi\rangle) + h_2(t) \right].$$

Comments: (i) For  $t = |\alpha|^2$  we get the LPS bound; i.e.  $f(|\alpha|^2) = 2[|\alpha|^2 E(\Psi) + |\beta|^2 E(\Phi) + h_2(|\alpha|^2)]$ .

(ii) Note that  $f(t) \geq |\alpha|^2 E(\Psi) + |\beta|^2 E(\Phi) + h_2(|\alpha|^2)$ . This is consistent with the case of biorthogonal states.

(iii) The minimum of the function  $f(t)$  is obtained at  $t = t^*$  where  $t^*$  satisfies the implicit equation:

$$\frac{|\alpha|^2(1-t^*)^2}{|\beta|^2(t^*)^2} = \frac{E(\Psi) - \log t^*}{E(\Phi) - \log(1-t^*)}.$$

(iv) Using the same idea presented in theorem 2, the upper bound in theorem 3 can be improved a bit by replacing  $h_2(t)$  in  $f(t)$  with  $h_2(t) - |S(\rho_t^A) - S(\rho_t^B)|$ .

(v) For the trivial case where  $\alpha = 1$  ( $\beta = 0$ ) we get  $f(t) = E(\Psi)$  for  $t = 1$ . That is, the upper bound equals  $E(\Gamma)$ . On the other hand, the LPS bound for  $\alpha = 1$  is  $2E(\Psi) = 2E(\Gamma)$ .

*Proof.* Consider the state

$$\rho_t^{AB} = t|\Psi\rangle\langle\Psi| + (1-t)|\Phi\rangle\langle\Phi|,$$

where  $0 \leq t \leq 1$ . For any  $\theta, \phi \in [0, 2\pi)$  we can construct a new decomposition of  $\rho_t^{AB}$ :

$$\rho_t^{AB} = q|\chi_1\rangle\langle\chi_1| + (1-q)|\chi_2\rangle\langle\chi_2| \quad (12)$$

where

$$\begin{aligned} \sqrt{q}|\chi_1\rangle &= \sqrt{t}\cos\theta|\Psi\rangle + \sqrt{1-t}e^{i\phi}\sin\theta|\Phi\rangle \\ \sqrt{1-q}|\chi_2\rangle &= -\sqrt{t}e^{-i\phi}\sin\theta|\Psi\rangle + \sqrt{1-t}\cos\theta|\Phi\rangle \\ q &= \|\sqrt{t}\cos\theta|\Psi\rangle + \sqrt{1-t}e^{i\phi}\sin\theta|\Phi\rangle\|. \end{aligned} \quad (13)$$

Now, from the properties of the von-Neumann entropy given in Eqs. (5,6), we deduce that

$$\begin{aligned} 0 &\leq S(\rho_t^A) - tE(|\Psi\rangle) - (1-t)E(|\Phi\rangle) \leq h_2(t) \\ 0 &\leq S(\rho_t^A) - qE(|\chi_1\rangle) - (1-q)E(|\chi_2\rangle) \leq h_2(q). \end{aligned}$$

From these inequalities we get

$$qE(|\chi_1\rangle) + (1-q)E(|\chi_2\rangle) \leq tE(|\Psi\rangle) + (1-t)E(|\Phi\rangle) + h_2(t). \quad (14)$$

Thus, since  $E(|\chi_2\rangle) \geq 0$  we find that

$$E(|\chi_1\rangle) \leq \frac{1}{q} \left[ tE(|\Psi\rangle) + (1-t)E(|\Phi\rangle) + h_2(t) \right]. \quad (15)$$

Note that so far we have 3 free parameters:  $t$ ,  $\theta$  and  $\phi$ . We now concentrate on all the convex decompositions of  $\rho_t^{AB}$  with  $|\chi_1\rangle = |\Gamma\rangle/\|\Gamma\|$ . This requirement reduces the number of free parameters to one and can be expressed in terms of the following conditions (see Eq. (13)):

$$\alpha' \equiv \frac{\alpha}{\|\Gamma\|} = \sqrt{\frac{t}{q}} \cos\theta \quad \text{and} \quad \beta' \equiv \frac{\beta}{\|\Gamma\|} = \sqrt{\frac{1-t}{q}} e^{i\phi} \sin\theta \quad (16)$$

Since  $|\Gamma\rangle = \alpha|\Psi\rangle + \beta|\Phi\rangle$  is defined up to a global phase we will assume, without loss of generality, that  $\alpha$  is real and non-negative. Similarly, we take  $\phi$  to be equal to the phase of  $\beta$  so that  $|\beta'| = \sqrt{(1-t)/q} \sin\theta$ . The parameter  $q$  can be written as a function of  $t$  and  $\theta$ . Note that

$$\frac{1}{\|\Gamma\|^2} = |\alpha'|^2 + |\beta'|^2 = \frac{t}{q} \cos^2\theta + \frac{1-t}{q} \sin^2\theta.$$

Hence,

$$\frac{q}{\|\Gamma\|^2} = t \cos^2\theta + (1-t) \sin^2\theta. \quad (17)$$

Now, substituting this form of  $q$  into Eq. (16) provides the relation between  $t$  and  $\theta$ :

$$\cos^2\theta = \frac{(1-t)|\alpha|^2}{t|\beta|^2 + (1-t)|\alpha|^2}; \quad \sin^2\theta = \frac{t|\beta|^2}{t|\beta|^2 + (1-t)|\alpha|^2}.$$

Finally, using these relations in eq. (17) gives

$$\frac{q}{\|\Gamma\|^2} = \frac{t(1-t)}{t|\beta|^2 + (1-t)|\alpha|^2}.$$

Hence, for decompositions with  $|\chi_1\rangle = |\Gamma\rangle/\|\Gamma\|$  we get (see Eq. (15))

$$\begin{aligned} \|\Gamma\|^2 E(|\Gamma\rangle) &\leq \frac{t|\beta|^2 + (1-t)|\alpha|^2}{t(1-t)} \\ &\quad \times \left[ tE(|\Psi\rangle) + (1-t)E(|\Phi\rangle) + h_2(t) \right]. \end{aligned}$$

for all  $0 < t < 1$ .  $\square$

*Example 3.* Here we consider an example where both Alice and Bob have Hilbert spaces of dimension  $d$ :

$$\begin{aligned} |\Psi\rangle &= \frac{1}{\sqrt{2}} \left( |11\rangle + \frac{1}{\sqrt{d-1}} [|22\rangle + |33\rangle + \dots + |dd\rangle] \right), \\ |\Phi\rangle &= \frac{1}{\sqrt{2}} \left( |11\rangle - \frac{1}{\sqrt{d-1}} [|22\rangle + |33\rangle + \dots + |dd\rangle] \right), \\ \alpha &= \frac{3}{5} \quad \text{and} \quad \beta = -\frac{4}{5}. \end{aligned} \quad (18)$$

This is the same example as one of the examples given in [1] except that here  $\alpha \neq -\beta$ . One can easily check that  $E(\Psi) = E(\Phi) = \frac{1}{2} \log(d-1) + 1$  and  $E(\alpha|\Psi) + \beta|\Phi\rangle) = (49/50) \log(d-1) + h_2(1/50)$ . Furthermore, it can be shown that in the limit  $d \rightarrow \infty$  the minimum of the function  $f(t)$  is obtained at  $t = 3/7$ . We therefore take this value to get an upper bound  $f(3/7) = (49/50) \log(d-1) + (49/25)h_2(3/7)$ . Thus, we have  $f(3/7) - E(\Gamma) = O(1)$ . On the other hand, for large  $d$  the LPS bound is approximately  $\log(d-1)$  and so we have  $\log(d-1) - E(\Gamma) \approx (1/50) \log(d-1) \rightarrow \infty$  as  $d \rightarrow \infty$ . That is, in the limit of high dimensions the LPS bound diverges from  $E(\Gamma)$  whereas the our bound approaches  $E(\Gamma)$ .

We know move to discuss lower bounds.

**Theorem 5.** *Let  $|\Psi\rangle$  and  $|\Phi\rangle$  be two bipartite states, and let  $|\Gamma\rangle = \alpha|\Psi\rangle + \beta|\Phi\rangle$  be a normalized state (i.e.  $\|\Gamma\| = 1$ ) for some  $\alpha, \beta \in \mathbb{C}$ . Then,*

$$E(|\Gamma\rangle) \geq \max\{L_1(t), L_2(t)\}, \quad (19)$$

for all  $0 \leq t \leq 1$ , where

$$L_1(t) = \frac{(1-t)|\beta|^2}{1-t(1-|\alpha|^2)} E(\Phi) - \frac{1-t}{t} E(\Psi) - \frac{1}{t} h_2(t)$$

$$L_2(t) = \frac{(1-t)|\alpha|^2}{1-t(1-|\beta|^2)} E(\Psi) - \frac{1-t}{t} E(\Phi) - \frac{1}{t} h_2(t).$$

Comments: (i) If  $|\Psi\rangle$  and  $|\Phi\rangle$  are orthogonal (i.e.  $|\alpha|^2 + |\beta|^2 = 1$ ) then we can obtain a simple lower bound by taking  $t = 1/(2|\alpha|^2)$  (or  $t = 1/(2|\beta|^2)$  if  $|\alpha|^2 < 1/2$ ):

$$E(|\Gamma\rangle) \geq (|\beta|^2 - |\alpha|^2) [E(\Phi) - E(\Psi)] - \frac{1}{|\gamma|^2} h_2(|\alpha|^2),$$

where  $|\gamma|^2 = \max\{|\alpha|^2, |\beta|^2\}$ . Note that in general  $t = 1/(2|\alpha|^2)$  does not maximize the function  $L_1(t)$  (or  $L_2(t)$ ) and therefore the bound above is just a simple bound and *not* the optimal one.

(ii) Note that it is inappropriate to replace  $h_2(t)$  in the theorem above with  $h_2(t) - |S(\rho_t^A) - S(\rho_t^B)|$  in order to improve it a bit. The reason is that this time  $\rho_t^A$  is a mixture that consists of  $|\Gamma\rangle$  itself.

(iii) For  $\alpha = 0$  (or  $\beta = 0$ ) the lower bound is  $E(\Gamma)$ . This gives us the first indication that the lower bound is tight.

(iv) The maximum of the function  $L_1(t)$  is obtained at  $t = t^*$  where  $t^*$  satisfies the implicit equation:

$$\frac{|\alpha|^2 |\beta|^2 (t^*)^2}{[1 - (1 - |\alpha|^2) t^*]^2} E(\Phi) = E(\Psi) - \log(1 - t^*). \quad (20)$$

Similar expression can be found for the value of  $t$  that maximizes  $L_2(t)$ .

*Proof.* Consider the state

$$\rho_t^{AB} = t|\Gamma\rangle\langle\Gamma| + (1-t)|\Psi\rangle\langle\Psi|.$$

where  $0 \leq t \leq 1$ . For any  $\theta, \phi \in [0, 2\pi)$  we can construct a new decomposition of  $\rho_t^{AB}$  just as in Eq.(12) except that now:

$$\begin{aligned} \sqrt{q}|\chi_1\rangle &= \sqrt{t} \cos \theta |\Gamma\rangle + \sqrt{1-t} e^{i\phi} \sin \theta |\Psi\rangle \\ \sqrt{1-q}|\chi_2\rangle &= -\sqrt{t} e^{-i\phi} \sin \theta |\Gamma\rangle + \sqrt{1-t} \cos \theta |\Psi\rangle \\ q &= \|\sqrt{t} \cos \theta |\Gamma\rangle + \sqrt{1-t} e^{i\phi} \sin \theta |\Psi\rangle\|^2. \end{aligned} \quad (21)$$

Using the same arguments as in Theorem 3 we find that:

$$qE(|\chi_1\rangle) + (1-q)E(|\chi_2\rangle) \leq tE(|\Gamma\rangle) + (1-t)E(|\Psi\rangle) + h_2(t).$$

Now, since  $E(|\chi_2\rangle) \geq 0$  we have

$$E(|\Gamma\rangle) \geq \frac{1}{t} [qE(|\chi_1\rangle) - (1-t)E(|\Psi\rangle) - h_2(t)]. \quad (22)$$

This equation holds for any choice of  $\theta$  and  $\phi$ . We would like now to choose  $\theta$  and  $\phi$  such that  $|\chi_1\rangle = |\Phi\rangle$ . From Eq.(21) we find that it is possible if

$$\sqrt{q} = \beta \sqrt{t} \cos \theta \quad \text{and} \quad \sqrt{t} \alpha \cos \theta = -\sqrt{1-t} e^{i\phi} \sin \theta.$$

Without loss of generality, we can assume that  $\beta$  is real (since  $|\Gamma\rangle$  is defined up to a global phase) and we take  $-e^{i\phi}$  to be the phase of  $\alpha$ . Furthermore, from these equations it follows that

$$q = \frac{t(1-t)|\beta|^2}{1-t(1-|\alpha|^2)}.$$

Substituting this value for  $q$  in Eq. (22) gives the lower bound  $L_1(t)$ . The lower bound  $L_2(t)$  is similarly obtained by exchanging the roles of  $|\Psi\rangle$  and  $|\Phi\rangle$ .  $\square$

In the following example we show that our lower bound can be very tight.

*Example 4.* Here we take  $|\Psi\rangle$ ,  $|\Phi\rangle$  and  $\alpha$  to be exactly the same as in Example 3, and  $\beta = 4/5$ . We therefore have  $E(\Psi) = E(\Phi) = \frac{1}{2} \log(d-1) + 1$  whereas  $E(\alpha|\Psi) + \beta|\Phi\rangle) = (1/50) \log(d-1) + h_2(1/50)$ . We would like now to find the value  $t = t^*$  in Eq. (20) at which  $L_1(t)$  is maximum. In the limit  $d \rightarrow \infty$  we can ignore the logarithmic term in Eq. (20) and so we get  $t^* = 25/28$ . The value of  $L_1(t)$  at  $t = 25/28$  is  $L_1(25/28) = (1/50) \log(d-1) + (1/25) - (28/25)h_2(25/28)$ . We therefore get that  $L_1(25/28)/E(\alpha|\Psi) + \beta|\Phi\rangle) \rightarrow 1$  at the limit  $d \rightarrow \infty$  and  $E(\alpha|\Psi) + \beta|\Phi\rangle) - L_1(25/28) \approx 0.65$ . The last value can be improved if one takes into account the logarithmic term in Eq. (20).

We end by making two observations. First, the lower bound given in Theorem 4 can also provide a lower bound on the entanglement of 2-dimensional bipartite subspaces [8] (see also [9] for the Schmidt rank of subspaces) by minimizing the bound over  $\alpha$  and  $\beta$ . This minimization also provides a lower bound on the entanglement of formation of a density matrix whose support

subspace is spanned by  $|\Psi\rangle$  and  $|\Phi\rangle$ . Second, in this Letter we have given lower and upper bounds for the entanglement of superpositions including two states. The question regarding the entanglement of superpositions with more than two terms is an important one for future work.

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